A side-by-side visual and algebraic proof of the Quadratic formula

Today, we will derive the quadratic formula algebraically and visually, for you to understand in both ways for unmatched clarity

What is the quadratic formula?

The quadratic formula is a very common formula in mathematics that allows for us to solve Quadratic Equations, which are in the form of

$$ax^2+bx+c=0$$

These equations can be solved using the Quadratic formula in a fairly painless manner. Today, we will work to derive it. The Quadratic Formula looks like this:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2}$$

The Derivation Of The Quadratic Formula

To start with the derivation, I will first take the general Quadratic Equation, which is as shown below. This part of the process will help convert the equation to a form we can represent in visual terms

$$ax^2+bx+c=0$$

We can simplify this equation by moving the “c” term over to other side of the equation, now, as we are moving it to the other side of the equation, its sign will be inverted, in this case, as it is being added on the left side of the equation, it will be subtracted on the right-hand side (The positive term will turn into a negative one). This now gives us the following equation:

$$ax^2+bx=-c$$

Now, we divide both sides of the equation by the “a” term, giving us the following equation:

$$\frac{ax^2}{a}+\frac{bx}{a}=\frac{-c}{a}$$

Here, dividing the “ax squared” term by the “a” term will make the “ a’s ” cancel out as we are first multiplying by them and then dividing by them.

This will then give us the following equation:

$$x^2+\frac{b}{a}x=\frac{-c}{a}$$

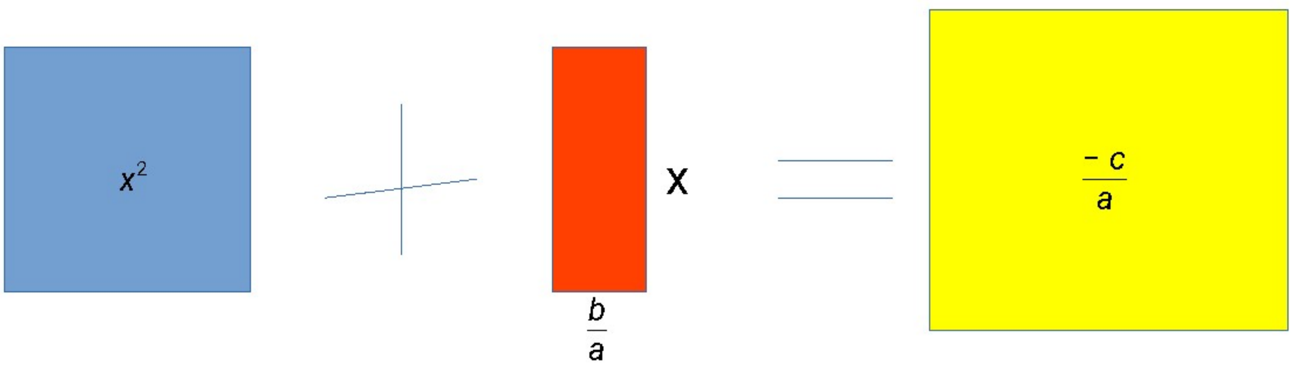
Now that we have converted this equation into a form that can be represented visually, we can start the side-by-side visual and algebraical process

Starting With The Visual Process

Now, we can set all these terms to be equal to the areas of shapes, so, the equation side by side with the shapes will look like this. We set x squared to equal the area of a square and set the second term; b over a times x to be the area of a rectangle with sides b over a and x respectively and the term negative "c" over a to be the area of a square so, our equation will look like this:

$$x^2+\frac{b}{a}x=\frac{-c}{a}$$

This can be visualised as the following:



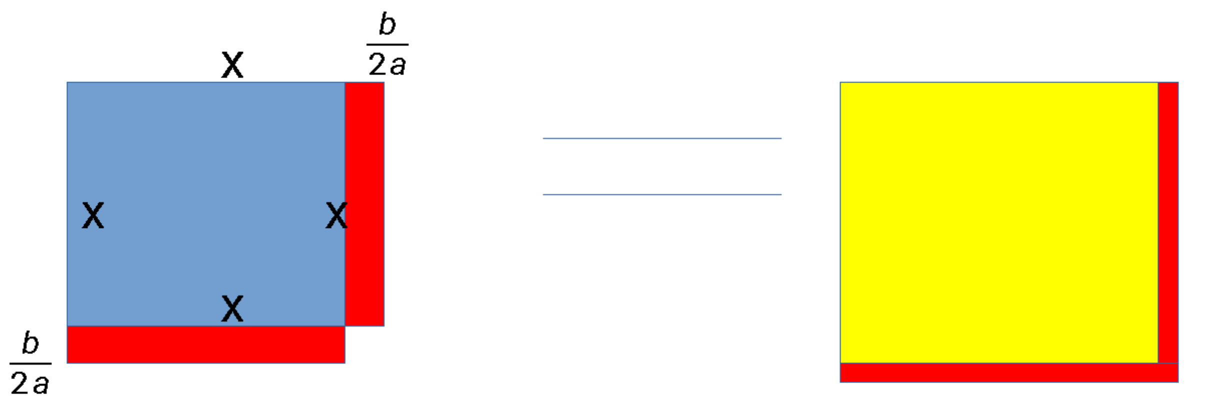
Here, you may think that how can a shape have negative area, so, I would like to warn you that, yes, it is not possible in real life, but, to prove this and many other theorems, we will regularly need to break math's connection to reality, so, please bear with me.

Now, here, the goal is to complete the blue square in order for it to equal the yellow square. Doing this will effectively be deriving the quadratic formula as

we equalise the sides of the equation. So, for the left side to be equal to the right side, we will have to figure out how to equalise the sides, and the first step in that would be to simplify the left side of the equation to find out what is the resulting shape to check the necessary steps to equalise the two sides of the equation. So, we half the “b over a” term to add to two sides of the blue square. Now, to divide by, we can not only divide by two, but also multiply by 1 over 2. This is shown below

$$\frac{\frac{b}{a}}{\frac{2}{1}} \text{ --> Reciprocal-->} \frac{b}{a} \times \frac{1}{2} = \frac{b}{2a}$$

So, half of the area of the rectangle is added to two perpendicular sides to be more of a square (to be equal to the yellow square) now, we have something like this:



Here, both the short sides of both the rectangles are equal to b over two a. Now, here we have a shape that is already very similar to a square, and, now, we shall complete it. The missing square that would complete the shape has the dimensions of b over 2a times b over 2a, as, the square’s sides are the same length as the short sides of the red rectangles, which have the length of b over 2a. Hence, the square’s dimensions will “b” over 2a squared, which is added also to the right-hand side to equalise the sides of the equations. Now,

the total area of the blue square is x plus b over two squared because the length of its sides is x plus b over two a. Its algebraical representation is

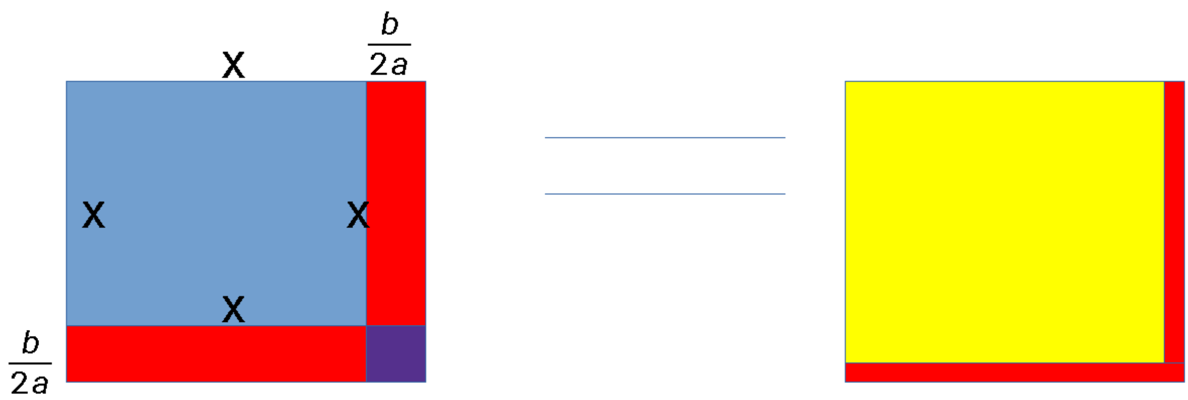
$$\left(x+\frac{b}{2a}\right)^2=\frac{-c}{a}+\left(\frac{b}{2a}\right)^2$$

Now, as we are adding a negative term, it is equivalent to subtraction (like adding debt instead of money)

Hence, the new algebraic representation will be this:

$$\left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

The visualisation is as follows:



Now, you may think that why did we not add the rectangles to the other side, well, it is very simple, because the original equation already said that the rectangles plus the blue square was equal to the original yellow square.

Now, from here on out, we will just simplify the formula we already have to get the quadratic formula.

So, the first step we will take to simplify it is to combine both the terms on the right-hand side to have common denominators (to be combined).

$$\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

Now, on simplifying the "b over 2a squared" term, we get the following

$$\frac{b^2}{4a^2} - \frac{c}{a}$$

Meaning, this is the new equation

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

Now, as 4a squared is simply (2 times 2) times (a times a), "a" is a factor of 4 a squared, hence, a common denominator can be simply found by dividing 4 a squared by a and multiplying the numerator and denominator by the quotient

$$\frac{2\times 2\times a\times a}{a}$$

This gives us one of the a's cancelling out, giving us 4a, hence, after multiplying the numerators and denominators by 4a, we get

$$\frac{-c\times 4a}{4a\times a} = \frac{-4ac}{4\times a\times a}$$

The "a times a" term turns into "a" squared, giving the equation this new form:

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Now, it is all a matter of algebraic simplification of this equation. Now, we take the square roots of both the sides of the equation, giving us the following equation:

$$\sqrt{\left(x+\frac{b}{2a}\right)^2}=\sqrt{\frac{b^2-4ac}{4a^2}}$$

The left side is being squared and then having the square root taken, hence, they are inverse operations and will cancel out.

As for the right side of the equation, the denominator's square root can be simplified to be 2a, this is because, as shown below, the denominator can be expanded to look like this

$4a^2$ -->Expanding--> $2\times 2\times a\times a$

So, as you can see, "4 a squared" is just 2 squared times a squared, so, as four's square root is 2 and "a squared" 's square root, by definition is "a", so, "4 a squared" 's square root is 2a, so, we now get this equation: (Also, the numerator's square root cannot be simplified)

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2}a$$

Now, you may think what is the reason for the “plus or minus” sign, but, when we talk about the square roots of positive integers. There are actually TWO SQUARE ROOTS OF POSITIVE INTEGERS. Yes, you must consider that two negative numbers, when multiplied, leads to a positive integer, so, take the square root of four, which is two, but if you take its negative, and multiply it by itself, then you will still get four, meaning that every positive integer, there are two square roots, one positive, and one is its negative. Hence, the plus or minus sign implies that there are two different roots taken, one positive and one negative, it effectively says either the negative or the positive square root. So, now, all we have to do is move the only last term apart from x still remaining, that is, b over 2a, which is positive on the left side, but will turn negative on the other side. The new equation is

$$x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

The denominators are also the same, so, they can be combined giving us the final quadratic formula: x = $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$